O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

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Referat

MAVZU:

BO'LAKLAB INTEGRALLASH VA RATSIONAL KASRLARNI INTEGRALLASH

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Toshkent-2015

BO’LAKLAB INTEGRALLASH

Bizga ikkita diferensiallanuvchi u(x) va v(x) funksiyalar berilgan .bo’lsin. Bu funksiyalar ko’paytmasi (uv) ning differensialini topaylik. Bu differensial quyidagicha aniqlanadi:

$$d(uv)=udv+vdu$$

Buni ikki tomonini hadma-had integrallab, qo’yidagini topamiz:

$$uv=\int udv+\int vdu \quad \text{yoki} \quad \int udv = uv-\int vdu \qquad (1)$$

Oxirgi topilgan ifoda bo’laklab integrallash formulasi deyiladi.

Bu formulani qo’llab integral hisoblaganda $\int udv$ ko’rinishdagi integral, ancha sodda bo’lgan $\int vdu$ ko’rinishdagi integralga keltiriladi.

Agar integral ostida $u=\ln x$ funksiya, yoki ikkita funksiyaning ko’paytmasi, hamda teskari trigonometrik funksiyalar qatnashgan bo’lsa,bunda bo’laklab integrallash formulasi qo’llaniladi. Bu usul bilan integrallanganda yangi o’zgaruvchiga o’tishning hojati yo’q.

Umuman aniqmas integralni hisoblaganda topilgan natija yoniga o’zgarmas (C=const) ni qo’shib qo’yish shart. Aks xolda integralning bitta qiymati topilib, qolganlari tashlab yuborilgan bo’ladi. Bu esa integrallashda xatolikka yo’l qo’yilgan deb hisoblanadi.

Misol. $\int x\,arctgx\,dx$ ni hisoblang.

$$u=arctgx \quad dv=xdx \quad du=\frac{dx}{1+x^2} \quad v=\int xdx = x^2/2$$

(bunda C=0 deb olindi)

(1) formulani qo’llaymiz

$$\int x\,arctgx\,dx=\frac{x^2}{2}arctgx-\int\frac{x^2}{2(1+x^2)}dx \qquad (*)$$

$$\int\frac{x^2}{1+x^2}dx$$

ni alohida hisoblaymiz

$$\int\frac{x^2}{1+x^2}dx=\int\frac{1+x^2-1}{1+x^2}dx=\int(1-\frac{1}{1+x^2})dx=x-arctgx+C$$

buni (*)ga qo’yamiz.

$$\int x\,arctgx\,dx=\frac{x^2}{2}arctgx-\frac{x}{2}+\frac{1}{2}arctgx+C=-\frac{x}{2}+\frac{x^2+1}{2}arctgx+C$$

RATSIONAL KASRLARNI INTEGRALLASH.

Ikkita ko'phadning nisbati ratsional kasr deyiladi. $\frac{P_n(x)}{Q_m(x)}$. Bunda $n<m$ bo'lsa ratsional kasr to'g'ri ratsional kasr deyiladi, $n \geq m$ bo'lsa, ratsional kasr noto'g'ri ratsional kasr deyiladi. Bunday kasr suratini maxrajiga bo'lish bilan butun va kasr qismlarga ajratiladi. Bundagi kasr to'g'ri kasr bo'ladi. Agar ratsional kasrda maxraj ya'ni $Q_m(x)=1$ bo'lsa, kasr butun ratsional funksiyaga aylanadi. Buni integrallash yuqorida ko'rib o'tilgan.

Endi to'g'ri ratsional kasrni integrallashni ko'rib o'tamiz. Avval oddiy ratsional kasrlarni integrallashni ko'ramiz. Umumiy holda ratsional kasr oddiy kasrlarga ajratilib, so'ngra integrallanadi. Oddiy ratsional kasrlar (ba'zan elementar kasrlar deb ham yuritiladi) qo'yidagi ko'rinishda bo'ladi.

1. $\frac{A}{x-a}$ 2. $\frac{A}{(x-a)^k}$ bu yerda $k \geq 2$ butun musbat son.

3. $\frac{Ax+B}{x^2+px+q}$. Maxrajni ildizi kompleks sonlardan iborat, ya'ni $\frac{p^2}{4}-q \leq 0$

4. $\frac{Ax+B}{(x^2+px+q)^k}$. $k \geq 2$ bo'lgan butun musbat son.

(1)-(4) ko'rinishdagi kasrlar eng sodda ratsional kasrlardir.

Endi shu kasrlarni integrallashni ko'raylik.

1. $\int \frac{A}{x-a}dx = A\ln|x-a|+C \qquad (C = const)$

2. $\int \frac{A}{(x-a)^k}dx = A\int (x-a)^{-k}d(x-a) =$

$$= A\frac{(x-a)^{-k+1}}{-k+1}+C = \frac{A}{(1-k)(x-a)^{k-1}}+C$$

3. $\int \frac{Ax+B}{x^2+px+q}dx = \int \frac{\frac{A}{2}(2x+p)+(B-\frac{Ap}{2})}{x^2+px+q}dx = \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx +$

$$+(B-\frac{Ap}{2})\int \frac{dx}{x^2+px+q} = \left[\begin{matrix} x^2+px+q=t \\ (2x+p)dx=dt \end{matrix} \qquad \int \frac{dt}{t} = \ln|t|+C\right] =$$

$$= \frac{A}{2}\ln|x^2+px+q|+(B-\frac{Ap}{2})\int \frac{dx}{(x+p/2)^2+(q-p^2/4)} =$$

$$= \frac{A}{2}\ln|x^2+px+q|+\frac{2B-Ap}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C$$

4. $\int \frac{Ax+B}{(x^2+px+q)^k}dx = \int \frac{A/2(2x+p)+B-Ap/2)}{(x^2+px+q)^k}dx =$

$$= \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}dx+(B-\frac{Ap}{2})\int \frac{dx}{(x^2+px+q)^k}.$$

Bu integrallardan birinchisi $x^2+px+q=t$ almashtirish bilan $\int \frac{dt}{t^k}$ ko'rinishdagi integralga keladi. Bu esa, integrallar jadvalidagi formulaga ko'ra $\frac{t^{-k+1}}{1-k}$ ga teng bo'ladi.

Ikkinchi integralning maxrajidan to'liq kvadrat ajratsak, hamda $x+p/2=t$ almashtirish bajarib va $q-p^2/4=m^2$ deb belgilasak, u holda $\int \frac{dt}{(t^2+m^2)^k}$ ko'rinishdagi integralga kelamiz. Bu integralni maxrajining darajasini qo'yidagicha ketma-ket kamaytirish bilan $\int \frac{dt}{t^2+m^2}$ ko'rinishdagi integralga keltiramiz. Bu esa, integrallar jadvalidagi formulaga binoan $\frac{1}{m} arctg \frac{t}{m}$ ga teng bo'ladi,
ya'ni

$$\int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2} \int \frac{(t^2+m^2)-t^2}{(t^2+m^2)^k} dt = \frac{1}{m^2} \int \frac{dt}{(t^2+m^2)^{k-1}} - \frac{1}{m^2} \int \frac{t^2}{(t^2+m^2)^k} dt \quad (*)$$

ammo, $\int \frac{t^2 dt}{(t^2+m^2)^k} = \int \frac{t \cdot tdt}{(t^2+m^2)^2} = -\frac{1}{2(k-1)} \int td\left(\frac{1}{(t^2+m^2)^{k-1}}\right)$

Buni bo'laklab integrallash formulasidan foydalanib qo'yidagi kurinishga keltiramiz:

$$\int \frac{t^2 dt}{(t^2+m^2)^k} = -\frac{1}{2(k-1)} \left[t \cdot \frac{1}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}} \right]$$

Buni (*) ga qo'yib qo'yidagini topamiz:

$$\int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2} \int \frac{dt}{(t^2+m^2)^{k-1}} + \frac{1}{m^2} \frac{1}{2(k-1)} \left[\frac{t}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}} \right] =$$

$$= \frac{1}{2m^2(k-1)(t^2+m^2)^{k-1}} + \frac{2k-3}{2m^2(k-1)} \int \frac{dt}{(t^2+m^2)^{k-1}}$$

o'ng tomondagi integral maxrajining daraja ko'rsatkichi bittaga kamaydi.

Shunday qilib, $\int \frac{dt}{(t^2+m^2)^k}$ integral maxrajining ko'rsatgichini bittaga kamaytirdik. Shu usul yordamida bu amalni takrorlash bilan berilgan integral $\int \frac{dt}{t^2+m^2}$ ko'rinishdagi integralga keltiriladi. To'rtinchi ko'rinishdagi ratsional kasrli integral shu yo'l bilan hisoblanadi.

Ko'rib o'tilgan ratsional kasrlar eng sodda (elementar) ratsional kasrlar edi. Endi boshqa ko'rinishdagi ratsional kasr berilgan bo'lsa,uni avval eng sodda ratsional kasrlar orqali ifodalab olib, keyin integrallash amali bajariladi. Shuning uchun ratsional kasrlarni elementar kasrlar orqali ifodalashni ko'ramiz. Qo'yidagi ko'rinishdagi to'g'ri kasrlarni har doim elementar kasrlar orqali ifodalash mumkin:

$\frac{A}{(x-a)^m}$; $\frac{Mx+N}{(x^2+px+q)^n}$. ($m$, n lar musbat butun sonlar)

Bizga to'g'ri $\frac{P_n(x)}{Q_m(x)}$ ratsional kasr berilgan bo'lsin. Bu kasr qo'yidagicha elementar kasrlarga ajratiladi:

a) $Q_m(x)$ maxraj ko'paytuvchilarga ajratiladi. Bunda chiziqli va kvadratik ko'paytuvchilar bo'lishi mumkin.

$$Q(x)=a_o(x-a_1)^m...(x-a_{m-k})^k(x^2+px+q)^n...(x^2+p_{n-r}x+q_{n-r})^r.$$

b) Berilgan kasrni elementar kasrlarga yoyilmasi qo'yidagi sxematik ko'rinishda bo'ladi:

$$\frac{P_n(x)}{Q_m(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+...+\frac{A_m}{(x-a)^m}+\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+$$
$$\frac{B_k}{(x-b)^k}+\frac{M_1x+N_1}{x^2+px+q}+\frac{M_2x+N_2}{(x^2+px+q)^2}+...+$$
$$+...+\frac{M_nx+N_n}{(x^2+px+q)^n}+...+\frac{C_1x+D_1}{x^2+cx+d}+...+\frac{C_rx+D_r}{(x^2+cx+d)^r} \quad (*)$$

bu yerda $A_m,...,B_k,...,M_n,...N_n,...,C_r,...,D_r$ lar o'zgarmas sonlar. $Q(x)$ necha karrali ildizga ega bo'lsa, sxemadagi elementar kasrlar soni shuncha bo'ladi.

v) Hosil bo'lgan (*) tenglikni har ikki tomonini $Q(x)$ ga ko'paytirish bilan kasrni maxrajdan qutqaramiz.

g) Keyin hosil bo'lgan tenglikni har ikki tomonidagi x ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, tenglamalar sistemasini hosil qilamiz.

Bu sistemadagi tenglamalar soni A_1, ..., B_1, ..., M_1, ..., N_1, ..., C_1,..., D_1, ..., noma'lumlar soniga teng bo'lishi kerak.

d) Hosil bo'lgan tenglamalar sistemasi yechilib, noma'lum koeffitsientlar topiladi va ular (*) ayniyatga qo'yiladi va ikki tomoni dx ga ko'paytirilib, integrallanadi. Hosil bo'lgan elementar kasrlar (1)-(4) ko'rinishdagi kasrlardan iborat bo'ladi.

Miso l.1. $\int\frac{3x^2+8}{x^3+4x^2+4x}dx$ hisoblansin.

Yechish. Integral ostidagi kasrning maxrajini ko'paytuvchilarga ajratamiz. $x^3+4x^2+4x=x(x^2+4x+4)=x(x+2)^2$

Endi berilgan kasrni (*) dan foydalanib, elementar kasrlarga yoyamiz:

$$\frac{3x^2+8}{x(x+2)^2}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{(x+2)^2} \qquad (**)$$

buni har ikki tomonini $x(x+2)^2$ ga ko'paytiramiz.

$$3x^2+8=A(x+2)^2+Bx(x+2)+Cx=(A+B)x^2+(4A+2B+C)x+4A$$

Endi x ni bir xil darajalari oldidagi koeffitsientlarini tenglashtirib, tenglamalar sistemasini tuzamiz:

$$\begin{cases} A+B=3 \\ 4A+2B+C=0 \\ 4A=8 \end{cases}$$

Bu sistemani yechib, A=2, B=1, C=-10 larni topamiz. So'ngra bularni (**) ga qo'yamiz.

$$\frac{3x^2+8}{x(x+2)^2}=\frac{2}{x}+\frac{1}{x+2}-\frac{10}{(x+2)^2}$$

Buni ikki tomonini dx ga ko'paytirib, keyin integrallaymiz:

$$\int\frac{3x^2+8}{x^3+4x^2+4x}dx=\int[\frac{2}{x}+\frac{1}{x+2}-\frac{10}{(x+2)^2}]dx=$$

$$=2\int\frac{dx}{x}+\int\frac{dx}{x+2}-10\int(x+2)^{-2}d(x+2)=$$

$$=2\ln|x|+\ln|x+2|+10/(x+2)+C$$

Misol 2. $\int\frac{x^3+4x^2-2x+1}{x^4+x}dx$ hisoblansin.

Yechish. Berilgan kasrni elementar kasrlarga ajratamiz. Buning uchun maxrajdagi ko'phadni ko'paytuvchilarga ajratamiz

$$x^4+x=x(x^3+1)=x(x+1)(x^2-x+1)$$

$$\frac{x^3+4x^2-2x+1}{x^4+x}=\frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}$$

$$x^3+4x^2-2x+1=A(x^3+1)+Bx(x^2-x+1)+(Cx+D)(x^2+x)=$$
$$=(A+B+C)x^3+(C+D-B)x^2+(B+D)x+A$$

Endi x larning bir xil darajalari oldidagi koeffitsientlarni tenglashtirish bilan noma'lum A, B, C, D larni aniqlash uchun qo'yidagi 4 ta tenglamani hosil qilamiz, hamda bu tenglamalar sistemasini yechib, A, B, C, D noma'lumlarni aniqlaymiz:

$$\begin{cases} A+B+C=1 & A=1 \\ C+D-B=4 & B=-2 \\ B+D=-2 & C=2 \\ A=1 & D=0 \end{cases}$$

Topilganlarni noma'lumlar o'rniga qo'yib kasrni elementar kasrlar orqali ifodasini yozamiz:

$$\frac{x^3+4x^2-2x+1}{x^4+x}=\frac{1}{x}-\frac{2}{x+1}+\frac{2x}{x^2-x+1}$$

endi buni integrallaymiz.

$$J=\int\frac{x^3+4x^2-2x+1}{x^4+x}dx=\int\frac{dx}{x}-2\int\frac{dx}{x+1}+2\int\frac{xdx}{x^2-x+1}=$$

$$=\ln|x|-2\ln|x+1|+2J_1$$

J_1 integralda x^2-x+1 dan to'la kvadrat ajratamiz:
$x^2-x+1=(x-1/2)^2+3/4$ Bunda $x-1/2=t$ $dx=dt$ deb olamiz. U holda J_1 qo'yidagicha hisoblanadi.

$$J_1=\int\frac{(t+1/2)dt}{t^2+3/4}=\frac{1}{2}\int\frac{2tdt}{t^2+3/4}+\frac{1}{2}\int\frac{dt}{t^2+3/4}=\frac{1}{2}\ln(t^2+3/4)+$$

$$+\frac{1}{\sqrt{3}}arctg\frac{2t}{\sqrt{3}}=\frac{1}{2}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}arctg\frac{2x-1}{\sqrt{3}}.$$

Natijada J integral qo'yidagicha aniqlanadi:

$$J=\ln\frac{|x|(x^2-x+1)}{(x+1)^2}+\frac{2}{\sqrt{3}}arctg\frac{2x-1}{\sqrt{3}}+C$$

Miso l.3. $\int\frac{(x^3-3)dx}{x^4+10x^2+25}$ hisoblansin.

Yechish. Maxrajni ko'paytuvchilarga ajratamiz:

$$x^4+10x^2+25=(x^2+5)^2$$

$$\frac{x^3-3}{x^4+10x^2+25}=\frac{Ax+B}{x^2+5}+\frac{Cx+D}{(x^2+5)^2}$$

$$x^3-3=(Ax+B)(x^2+5)+Cx+D=Ax^3+Bx^2+(5A+C)x+(5B+D)$$

$$\begin{cases} A=1 & A=1 \\ B=0 & B=0 \\ 5A+C=0 & C=-5 \\ 5B+D=-3 & D=-3 \end{cases}$$

$$\frac{x^3-3}{x^4+10x^2+25}=\frac{x}{x^2+5}-\frac{5x+3}{(x^2+5)^2}$$

Buni integrallaymiz:

$$J=\int\frac{x^3-3}{x^4+10x^2+25}dx=\underbrace{\int\frac{xdx}{x^2+5}}_{J_1}-5\underbrace{\int\frac{xdx}{(x^2+5)^2}}_{J_2}-3\underbrace{\int\frac{dx}{(x^2+5)^2}}_{J_3}.$$

$$J_1=\int\frac{xdx}{x^2+5}=\frac{1}{2}\int\frac{2xdx}{x^2+5}=\frac{1}{2}\int\frac{d(x^2+5)}{x^2+5}=\frac{1}{2}\ln(x^2+5)$$

$$J_2=\int\frac{xdx}{(x^2+5)^2}=\frac{1}{2}\int(x^2+5)^{-2}d(x^2+5)=\frac{1}{2}\frac{(x^2+5)^{-1}}{-1}=-\frac{1}{2(x^2+5)}$$

J_3 integralda o'zgaruvchini almashtiramiz:

$x=\sqrt{5}tgz dx=\sqrt{5}\sec^2 zdz$

$$J_3=\int\frac{dx}{(x^2+5)^2}=\int\frac{\sqrt{5}\sec^2 zdz}{25\sec^4 z}=\frac{1}{5\sqrt{5}}\int\cos^2 zdz=$$

$$=\frac{1}{10\sqrt{5}}\int(1+\cos 2z)dz=\frac{1}{10\sqrt{5}}(z+\frac{1}{2}\sin 2z)=\frac{1}{10\sqrt{5}}(arctg\frac{x}{\sqrt{5}}+\frac{x\sqrt{5}}{x^2+5}).$$

Berilgan integral qo'yidagiga teng bo'ladi:

$$J=\frac{1}{2}\ln(x^2+5)+\frac{5}{2(x^2+5)}-\frac{3}{10\sqrt{5}}(arctg\frac{x}{\sqrt{5}}+\frac{x\sqrt{5}}{x^2+5})+C=$$

$$=\frac{1}{2}\ln(x^2+5)+\frac{25-3x}{10(x^2+5)}-\frac{3}{10\sqrt{5}}arctg\frac{x}{\sqrt{5}}+C.$$