O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

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Referat

MAVZU:

DARAJALI QATORLAR. QATORLAR YORDAMIDA TAQRIBIY HISOBLASHLAR.

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DARAJALI QATORLAR. QATORLAR YORDAMIDA TAQRIBIY HISOBLASHLAR.

Ta'rif. Hadlari darajali funksiyalardan iborat bo'lgan

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots = \sum_{n=0}^{\infty} a_n x^n \qquad (1)$$

ko'rinishdagi qator darajali qator deb ataladi.

Abel teoremasi. 1) Agar darajali qator $x=x_0\neq 0$ nuqtada yaqinlashsa, u holda bu qator $-|x_0|<x<|x_0|$ oraliqda absolyut yaqinlashadi; 2) Agar darajali qator $x=x_0^{|}$ nuqtada uzoqlashsa, u holda bu qator $-|x_0^{|}|>x$ va $x>|x_0^{|}|$ oraliqlarda uzoqlashadi;

Isboti. 1) Teoremaning shartiga ko'ra

$$a_0+a_1x_0+a_2x_0^2+\ldots+a_nx_0^n+\ldots \qquad (2)$$

qator yaqinlashadi, demak $n\to\infty$ da $a_nx_0^n\to 0$, bu degani shunday bir musbat M soni mavjud bo'ldiki, qatorning hamma hadi absolyut qiymati bo'yicha M dan kichik bo'ladi. (1) qatorni

$$a_0 + a_1 x_0\left(\frac{x}{x_0}\right) + a_2 x_0^2\left(\frac{x}{x_0}\right)^2 + \ldots + a_n x_0^n\left(\frac{x}{x_0}\right)^n + \ldots \qquad (3)$$

ko'rinishda yozib olamiz va

$$|a_0| + |a_1 x_0|\left|\frac{x}{x_0}\right| + |a_2 x_0^2|\left|\frac{x}{x_0}\right|^2 + \ldots + |a_n x_0^n|\left|\frac{x}{x_0}\right|^n + \ldots \qquad (4)$$

qatorni ko'raylik. Bu qatorning hadlari

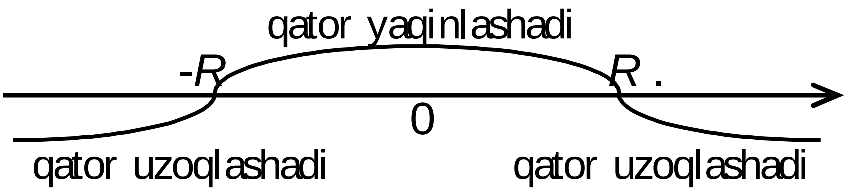
$$M + M\left|\frac{x}{x_0}\right| + M\left|\frac{x}{x_0}\right|^2 + \ldots + M\left|\frac{x}{x_0}\right|^n + \ldots \qquad (5)$$

qatorning mos hadidan kichik. $|x|<|x_0|$ tengsizlik bajarilganda (5) qator maxraji $q=\left|\frac{x}{x_0}\right|<1$ ga teng bo'lgan cheksiz kamayuvchi geometrik progressiyani tashkil etadi, demak, yaqinlashadi. Shunday qilib, (5) qator yaqinlashgani uchun (4) qator ham yaqinlashadi, natijada (3) qator yoki (1) qator absolyut yaqinlashadi.

2) Endi teoremaning ikkinchi qismini ham isbot qilish unchalik qiyin emas: faraz qilaylik x_0^1 nuqtada (1) qator uzoqlashsin. U holda $|x|>|x_0^{|}|$

tengsizlikni qanoatlantiruvchi har qanday x nuqtada ham qator uzoqlashadi. Demak, $-|x_0|>x$ va $x>|x_0|$ oraliqlarda (1) qator uzoqlashadi. Shunday qilib, teorema to’la isbot qilindi.

Ta’rif. Darajali qatorning yaqinlashish sohasi markazi koordinat boshida yotadigan intervaldan iboratdir. Darajali qatorning yaqinlashish intervali deb shunday $-R$ dan $+R$ gacha bo’lgan intervalga aytiladiki, bu intervalning ichida yotadigan har qanday x nuqtada qator absolyut yaqinlashadi, intervalning tashqarisida yotadigan istalgan x nuqtada esa uzoqlashadi.



R soni darajali qatorning yaqinlashish radiusi deb aytiladi. Intervalning oxirlarida (ya’ni $x=-R$ va $x=R$ nuqtalarida) berilgan qatorning yaqinlashishi va uzoqlashishi haqidagi savol har bir qator uchun alohida yechiladi.

1. Darajali qatorning yaqinlashish intervalini (yoki yaqinlashish radiusini) topish.

(1) darajali qatorni

$$|a_0|+|a_1||x|+|a_2||x|^2+|a_3||x|^3+\ldots+|a_n||x|^n+\ldots \qquad (6)$$

ko’rinishda yozib olamiz. Bu qator musbat hadli qator bo’lgani uchun uning yaqinlashishini Dalamber alomatiga ko’ra aniqlaymiz. Faraz qilaylik, quyidagi limit mavjud bo’lsin:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x|=L|x|$$

Unda agar $L|x|<1$ bo’lsa, ya’ni $|x|<1/L$ yoki $-1/L<x<1/L$ intervalda qator absolyut yaqinlashadi.

Agar $L|x|>1$ bo’lsa, ya’ni $|x|>1/L$ yoki $-1/L>x$ va $x>1/L$ intervallarda qator uzoqlashadi. Yaqinlashish radiusi

$$R=\frac{1}{L}=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

formulaga ko’ra topiladi. Shunga o’xshab R ni Koshi alomatini qo’llab ham topish mumkin: $R=\dfrac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}$;

Misol. $\frac{x}{2}+\frac{2^2x^2}{2^2}+\frac{3^2x^3}{2^3}+\frac{4^2x^4}{2^4}+\ldots+\frac{n^2x^n}{2^n}+\ldots$ darajali qatorning yaqinlashish intervali topilsin.

Yechish. Bu yerda $a_n=\frac{n^2}{2^n}$, $a_{n+1}=\frac{(n+1)^2}{2^{n+1}}$, demak,

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{n^2\,2^{n+1}}{2^n(n+1)^2}=2$$

Javob. Berilgan darajali qatorning yaqinlashish intervali $-2<x<2$ tengsizlikdan iborat. Intervalning chegaralarida qator uzoqlashadi.

2. Teylor va Makloren qatorlari.

Agar $y=f(x)$ funksiya $x=a$ nuqtaning atrofida $(n+1)$-nchi tartibgacha hosilaga ega bo'lsa Teylor formulasi deb ataluvchi

$$f(x)=f(a)+\frac{x-a}{1!}f^{\prime}(a)+\frac{(x-a)^2}{2!}f^{\prime\prime}(a)+\ldots+$$
$$+\frac{(x-a)^n}{n!}f^{(n)}(a)+R_n(x). \qquad (1)$$

formula bizga ma'lum, bu yerda

$$R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)]$$

qoldiq had edi, $0<\theta<1$.

Agar $f(x)$ funksiya $x=a$ nuqtaning atrofida istalgan tartibgacha hosilaga ega bo'lsa, Teylor formulasidagi n istalgancha katta qilib olinishi mumkin. Faraz qilaylik $\lim_{n\to\infty}R_n(x)=0$ bajarilsin, u holda (1) formulada $n\to\infty$ da limitga o'tib, o'ng tomonda qator hosil qilinadi va u Teylor qatori deb ataladi:

$$f(x)=f(a)+\frac{x-a}{1!}f^{\prime}(a)+\frac{(x-a)^2}{2!}f^{\prime\prime}(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots \qquad (2)$$

(2) tenglik $\lim_{n\to\infty}R_n(x)\to 0$ bajarilgandagina o'rinlidir.

Agar Teylor qatorida $a=0$ desak uning xususiy ko'rinishi bo'lgan Makloren qatori hosil bo'ladi:

$$f(x)=f(0)+\frac{x}{1!}f^{|}(0)+\frac{x^2}{2!}f^{||}(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots \qquad (3)$$

Berilgan $f(x)$ funksiyani Teylor qatoriga yoyish uchun:
a) $f(x)$ funksiyaning barcha tartibdagi hosilalarining $x=a$ nuqtadagi qiymatlari hisoblanadi va Teylor qatorining yoyilmasiga olib borib qo'yiladi;
b) hosil bo'lgan qatorning yaqinlashish sohasi topiladi.

Misol. $f(x)=2^x$ funksiya x ning darajalari bo'yicha Teylor qatoriga yoyilsin.

Yechish. a) 2^x funksiyaning barcha tartibdagi hosilalarini x=0 nuqtadagi qiymatlarini topamiz:

$f(x)=2^x$, $\qquad f(0)=1$;
$f^{|}(x)=2^x\ln2$, $\qquad f^{|}(0)=\ln2$;
$f^{||}(x)=2^x\ln^2 2$, $\qquad f^{||}(0)=\ln^2 2$;
....
$f^{(n)}(x)=2^x\ln^n 2$, $\qquad f^{(n)}(0)=\ln^n 2$;
....

Endi topilgan qiymatlarni (3) ifodaga qo'yib, 2^x funksiya uchun x ning darajalari bo'yicha Teylor qatorini hosil qilamiz:

$$2^x=1+\frac{\ln 2}{1!}x+\frac{\ln^2 2}{2!}x^2+\ldots+\frac{\ln^n 2}{n!}x^n+\ldots$$

b) hosil bo'lgan qatorning yaqinlashish sohasini topamiz:

$R=\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim\limits_{n\to\infty}\frac{\ln^n 2(n+1)!}{n!\ln^{n+1}2}=\infty$ dan ko'rinadiki topilgan qator x ning har qanday qiymatlarida yaqinlashadi.

3. Asosiy funksiyalar yoyilmasining jadvali:

1. $e^x=\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots \qquad (|x|<\infty);$

$$2.\ \sin x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\quad (|x|<\infty);$$

$$3.\ \cos x=\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\quad (|x|<\infty);$$

$$4.\ \frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+x^4+\ldots\qquad (|x|<1);$$

$$5.\ (1+x)^m=1+\sum_{n=1}^{\infty}\frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n=$$

$$=1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\ldots\ (|x|<1)$$

(binomial m-istalgan haqiqiy son);

$$6.\ \ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n}}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\quad (-1<x\le 1);$$

$$7.\ arctgx=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\ (|x|\le 1);$$

4. Qatorlar yordamida taqribiy hisoblashlar.

$f(x)$ funksiyaning x_0 nuqtadagi qiymatini taqribiy hisoblash uchun bu funksiya darajali qatorga yoyiladi va yoyilmadagi x lar o’rniga x_0 qiymat qo’yiladi. Shundan keyin $f(x_0)$ qiymatni kerakli aniqlikda hisoblash uchun qatorning zarur sondagi boshlang’ich hadlari olinadi. Masalan, arcsin1/10 ni hisoblash uchun arcsin$x$ funksiyani darajali qatorga yoyish (x ning darajalari bo’yicha) va undagi x lar o’rniga 1/10 qiymatni qo’yish kerak.

4-misol. $\sqrt[4]{e}$ 0,00001 aniqlikda hisoblansin.

Yechilishi. $e^x\approx 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+$; yoyilmada x=1/4 deb olamiz:

$$e^{1/4}\approx 1+\frac{1}{4}+\frac{1}{4^2\cdot 2!}+\frac{1}{4^3\cdot 3!}+\frac{1}{4^4\cdot 4!}+\ldots$$

Ushbu hisoblashda $|x|<n+1$ lar uchun qilinadigan xatolik $|R_n|< \frac{|x|^{n+1}}{n!(n+1-|x|)}$

tengsizlikdan topiladi.

Agar n=4 deb, beshta hadni olsak ko'rilayotgan hisoblashdagi xatolik 0,00001 dan oshmaydi:

$$R_n < \frac{x^{4+1}}{4!(4+1-x)} = \frac{1}{4^5 \cdot 4!(5-1/4)} < 0{,}00001$$

5-misol. $\cos 1^{o}$ 0,0001 aniqlikda hisoblansin.

Yechilishi. Kosinusning taqribiy qiymatlarini hisoblashda

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^n \frac{x^{2n}}{(2n)!}$$

formuladan foydalaniladi.

Bunda qilinadigan xatolik

$$|R_{2n}| \le \frac{x^{2n+2}}{(2n+2)!}$$

tengsizlikdan topiladi.

Demak, $\cos 1^0 = \cos\frac{\pi}{180}$ bo'lgani uchun kosinusning yoyilmasida $x = \frac{\pi}{180}$ deb birinchi ikkita hadni olsak,

$$\cos 1^0 \approx 1 - \frac{\pi^2}{180^2 \cdot 2!} \approx 0{,}9998$$

hosil bo'ladi. Bunda qilingan xatolik nihoyatda kichikdir:

$$|R_2| \le \frac{\pi^4}{180^4 \cdot 4!} < \frac{4^4}{180^4 \cdot 4!} = \frac{1}{45^2 \cdot 24} < 0{,}0000001.$$